

HÖLDER REGULARITY FOR SOLUTIONS TO COMPLEX MONGE-AMPÈRE EQUATIONS

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ABSTRACT. We consider the Dirichlet problem for the complex Monge-Ampère equation in a bounded strongly hyperconvex Lipschitz domain in \mathbb{C}^n . We first give a sharp estimate on the modulus of continuity of the solution when the boundary data is continuous and the right hand side has a continuous density. Then we consider the case when the boundary value function is $C^{1,1}$ and the right hand side has a density in $L^p(\Omega)$ for some $p > 1$ and prove the Hölder continuity of the solution.

1. INTRODUCTION

Let Ω be a bounded pseudoconvex domain in \mathbb{C}^n . Given $\varphi \in \mathcal{C}(\partial\Omega)$ and $0 \leq f \in L^1(\Omega)$. We consider the Dirichlet problem:

$$Dir(\Omega, \varphi, f) : \begin{cases} u \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega}) \\ (dd^c u)^n = f \beta^n \\ u = \varphi \end{cases} \quad \begin{matrix} \text{in } \Omega \\ \text{on } \partial\Omega \end{matrix}$$

where $PSH(\Omega)$ is the set of plurisubharmonic (psh) functions in Ω . Here we denote $d = \partial + \bar{\partial}$ and $d^c = i/4(\bar{\partial} - \partial)$ then $dd^c = i/2\partial\bar{\partial}$ and $(dd^c \cdot)^n$ stands for the complex Monge-Ampère operator.

If $u \in \mathcal{C}^2(\Omega)$ and is plurisubharmonic function, the complex Monge-Ampère operator is given by

$$(dd^c u)^n = \det \left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right) \beta^n$$

where $\beta = i/2 \sum_{j=1}^n dz_j \wedge d\bar{z}_j$ be the standard Kähler form in \mathbb{C}^n .

In their seminal work, Bedford and Taylor proved that the complex Monge-Ampère operator can be extended to the set of bounded plurisubharmonic functions (see [BT76], [BT82]). Moreover, it is invariant under holomorphic change of coordinates. We refer the reader to [BT76], [De89], [Kl91], [Ko05] for more details on its properties.

This problem has been studied extensively in last decades by many authors. When Ω is a bounded strongly pseudoconvex domain with smooth boundary, Bedford and Taylor had showed that $Dir(\Omega, \varphi, f)$ has a unique continuous solution $U := U(\Omega, \varphi, f)$. Furthermore, it was proved in [BT76] that $U \in Lip_\alpha(\bar{\Omega})$ when $\varphi \in Lip_{2\alpha}(\partial\Omega)$ and $f^{1/n} \in Lip_\alpha(\bar{\Omega})$ ($0 < \alpha \leq 1$). In the non degenerate case i.e. $0 < f \in \mathcal{C}^\infty(\bar{\Omega})$ and $\varphi \in \mathcal{C}^\infty(\partial\Omega)$, Caffarelli,

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Kohn, Nirenberg and Spruck proved in [CKNS85] that $U \in C^\infty(\bar{\Omega})$. However a simple example of Gamelin and Sibony shows that the solution is not, in general, better than $C^{1,1}$ -smooth when $f \geq 0$ and smooth ([GS80]). Krylov proved that if $\varphi \in C^{3,1}(\partial\Omega)$ and $f^{1/n} \in C^{1,1}(\bar{\Omega})$, $f \geq 0$ then $U \in C^{1,1}(\bar{\Omega})$ (see [Kr89]).

For B -regular domains, Blocki [Bl96] proved the existence of a continuous solution to the Dirichlet problem $Dir(\Omega, \varphi, f)$ when $f \in C(\bar{\Omega})$.

For a strongly pseudoconvex domain with smooth boundary, Kołodziej demonstrated in [Ko98] that $Dir(\Omega, \varphi, f)$ still admit a unique continuous solution under the milder assumption $f \in L^p(\Omega)$, for $p > 1$. Recently Guedj, Kołodziej and Zeriahi studied the Hölder continuity of the solution when $0 \leq f \in L^p(\Omega)$, for some $p > 1$, is bounded near the boundary (see [GKZ08]).

For the complex Monge-Ampère equation on a compact Kähler manifold, Hölder continuity of the solution was proved earlier by Kołodziej [Ko08] (see also [DDGHKZ12]).

A viscosity approach to the complex Monge-Ampère equation has been developed in [EGZ11] and [Wan12].

In this paper, we consider the more general case where Ω be a bounded strongly hyperconvex Lipschitz domain (the boundary does not need to be smooth) and $f \in L^p(\Omega)$.

We will generalize the approach of Bedford and Taylor [BT76] by showing an estimate for the modulus of continuity to the solution in terms of the modulus of continuity of the data.

Theorem A. *Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly hyperconvex Lipschitz domain, $\varphi \in C(\partial\Omega)$ and $0 \leq f \in C(\bar{\Omega})$. Assume that ω_φ is the modulus of continuity of φ and $\omega_{f^{1/n}}$ is the modulus of continuity of $f^{1/n}$. Then the modulus of continuity of U has the following estimate*

$$\omega_U(t) \leq \eta(1 + \|f\|_{L^\infty(\bar{\Omega})}^{1/n}) \max\{\omega_\varphi(t^{1/2}), \omega_{f^{1/n}}(t), t^{1/2}\}$$

where η is a positive constant depending on Ω .

Here we will use a new description of the solution given by Proposition 3.3 to get an optimal control for the modulus of continuity of this solution in a strongly hyperconvex Lipschitz domain.

For more general density $f \in L^p(\Omega)$ for some $p > 1$, it was shown in [GKZ08] that the unique solution to $Dir(\Omega, \varphi, f)$ belongs to $C^{0,\alpha}(\bar{\Omega})$ for all $\alpha < 2/(nq+1)$ when $\varphi \in C^{1,1}(\partial\Omega)$ and $f \in L^p(\Omega)$ be a bounded function near the boundary. Here we will improve this result and show the following theorem

Theorem B. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded strongly hyperconvex Lipschitz domain. Assume that $\varphi \in C^{1,1}(\partial\Omega)$ and $f \in L^p(\Omega)$ for some $p > 1$. Then the unique solution U to $Dir(\Omega, \varphi, f)$ is α -Hölder continuous on $\bar{\Omega}$ for any $0 < \alpha < 1/(nq+1)$ where $1/p + 1/q = 1$. Moreover, if $p \geq 2$, then the solution U is α -Hölder continuous on $\bar{\Omega}$ for any $0 < \alpha < \min\{1/2, 2/(nq+1)\}$.*

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2. PRELIMINARIES

We recall that a hyperconvex domain is to be a domain in \mathbb{C}^n admitting a bounded exhaustion function.

Let us define the class of hyperconvex domains which will be considered in this paper.

Definition 2.1. *A bounded domain $\Omega \subset \mathbb{C}^n$ is called strongly hyperconvex Lipschitz (shortly SHL) domain if there exists a neighbourhood Ω' of $\bar{\Omega}$ and a Lipschitz plurisubharmonic function $\rho : \Omega' \rightarrow \mathbb{R}$ such that*

- (1) $\rho < 0$ in Ω and $\partial\Omega = \{\rho = 0\}$,
- (2) *there exists a constant $c > 0$ such that $dd^c\rho \geq c\beta$ in Ω in the weak sense of currents.*

Example 2.2.

- (1) *Let Ω be a strictly convex domain that is there exists a Lipschitz defining function ρ such that $\rho - c|z|^2$ is convex for some $c > 0$. It is clear that Ω is strongly hyperconvex Lipschitz domain.*
- (2) *A smooth strictly pseudoconvex bounded domain is a SHL domain (see [HL84]).*
- (3) *The nonempty finite intersection of strictly pseudoconvex bounded domains with smooth boundary in \mathbb{C}^n is a bounded SHL domain. In fact, it is sufficient to put $\rho = \max\{\rho_i\}$. More generally a finite intersection of SHL domains is an SHL domain.*
- (4) *The domain $\Omega = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n; |z_1| + \dots + |z_n| < 1\}$ ($n \geq 2$) is a bounded strongly hyperconvex Lipschitz domain in \mathbb{C}^n with non smooth boundary.*
- (5) *The unit polydisc in \mathbb{C}^n ($n \geq 2$) is hyperconvex with Lipschitz boundary but it is not a strongly hyperconvex Lipschitz.*

Remark 2.3. *Kerzman and Rosay [KR81] proved that in a hyperconvex domain there exists there exists an exhaustion function which is smooth and strictly plurisubharmonic. Furthermore, they proved that any bounded pseudoconvex domain with C^1 -boundary is hyperconvex domain. This result was extended by Demailly [De87] to bounded pseudoconvex domains with Lipschitz boundary.*

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. If $u \in PSH(\Omega)$ then $dd^c u \geq 0$ in the sense of currents. We define

$$(2.1) \quad \Delta_H u := \sum_{j,k=1}^n h_{j\bar{k}} \frac{\partial^2 u}{\partial z_k \partial \bar{z}_j}$$

for every positive definite Hermitian matrix $H = (h_{j\bar{k}})$. We can see $\Delta_H u$ as a positive Radon measure in Ω .

The following lemma is elementary and important for the sequel (see [Gav77]).

Lemma 2.4. ([Gav77]). *Let Q be a $n \times n$ nonnegative hermitian matrix. Then*

$$(\det Q)^{\frac{1}{n}} = \inf\{tr(H.Q); H \in H_n^+ \text{ and } \det(H) = n^{-n}\}$$

where H_n^+ denotes the set of all positive hermitian $n \times n$ matrices.

Example 2.5. We calculate $\Delta_H(|z|^2)$ for every matrix $H \in H_n^+$ and $\det H = n^{-n}$.

$$\Delta_H(|z|^2) = \sum_{j,k=1}^n h_{j\bar{k}} \cdot \delta_{k\bar{j}} = tr(H)$$

using the inequality of arithmetic and geometric means, we have :

$$1 = (\det I)^{\frac{1}{n}} \leq tr(H),$$

hence $\Delta_H(|z|^2) \geq 1$ for every matrix $H \in H_n^+$ and $\det(H) = n^{-n}$.

Using ideas from the theory of viscosity due to Eyssidieux, Guedj and Zeriahi [EGZ11], we can prove the following result.

Proposition 2.6. Let $u \in PSH \cap L^\infty(\Omega)$ and $0 \leq f \in \mathcal{C}(\bar{\Omega})$. Then the following conditions are equivalent:

- 1) $\Delta_H u \geq f^{1/n}$ in the weak sense of distributions, for any $H \in H_n^+$ and $\det H = n^{-n}$.
- 2) $(dd^c u)^n \geq f \beta^n$ in the weak sense of currents in Ω .

This result is implicitly contained in [EGZ11], but we will give a complete proof for the convenience of the reader.

Proof. First, suppose that $u \in \mathcal{C}^2(\Omega)$, then by Lemma 2.4 the following

$$\Delta_H u = \sum_{j,k=1}^n h^{j\bar{k}} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \geq f^{1/n}, \forall H \in H_n^+, \det(H) = n^{-n}$$

is equivalent to

$$\left(\det \left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right) \right)^{1/n} \geq f^{1/n}.$$

The last inequality means that

$$(dd^c u)^n \geq f \beta^n.$$

(1 \Rightarrow 2) Let (ρ_ϵ) be a family of regularizing kernels with $\text{supp } \rho_\epsilon \subset B(0, \epsilon)$ and $\int_{B(0, \epsilon)} \rho_\epsilon = 1$, hence the sequence $u_\epsilon = u * \rho_\epsilon$ decreases to u , then we see that (1) implies $\Delta_H u_\epsilon \geq (f^{1/n})_\epsilon$. Since u_ϵ is smooth, we use the first case and get $(dd^c u_\epsilon)^n \geq ((f^{1/n})_\epsilon)^n \beta^n$, hence by applying the convergence theorem of Bedford and Taylor (Theorem 7.4 in [BT82]) we obtain $(dd^c u)^n \geq f \beta^n$.

(2 \Rightarrow 1) Fix $x_0 \in \Omega$, and q is \mathcal{C}^2 -function in a neighborhood B of x_0 such that $u \leq q$ in this neighborhood and $u(x_0) = q(x_0)$.

First step: we will show that $dd^c q_{x_0} \geq 0$. Indeed, for every small enough ball $B' \subset B$ centered at x_0 , we have

$$u(x_0) - q(x_0) \geq \frac{1}{V(B')} \int_{B'} (u - q) dV,$$

then we get

$$\frac{1}{V(B')} \int_{B'} q dV - q(x_0) \geq \frac{1}{V(B')} \int_{B'} u dV - u(x_0) \geq 0.$$

Since q is \mathcal{C}^2 -smooth and the radius of B' tend to 0, it follows, from Proposition 3.2.10 in [H94], that $\Delta q_{x_0} \geq 0$. For every positive definite Hermitian matrix H with $\det H = n^{-n}$, we make linear change of complex coordinates T such that H will be I (the identity matrix) in the new coordinates and $\tilde{Q} = (\partial^2 \tilde{q} / \partial w_j \partial \bar{w}_k)$ where $\tilde{q} = q \circ T^{-1}$ then

$$\Delta_H q(x_0) = \text{tr}(H \cdot Q) = \text{tr}(I \cdot \tilde{Q}) = \Delta \tilde{q}(y_0)$$

Hence $\Delta_H q(x_0) \geq 0$ for every $H \in H_n^+$ then $dd^c q_{x_0} \geq 0$.

Second step: we claim that $(dd^c q)_{x_0}^n \geq f(x_0)\beta^n$. Suppose that there exists a point $x_0 \in \Omega$ and a \mathcal{C}^2 -function q which satisfies $u \leq q$ in a neighborhood of x_0 and $u(x_0) = q(x_0)$ such that $(dd^c q)_{x_0}^n < f(x_0)\beta^n$. we put

$$q^\epsilon(x) = q(x) + \epsilon \left(\|x - x_0\|^2 - \frac{r^2}{2} \right)$$

for $0 < \epsilon \ll 1$ small enough, we see that

$$0 < (dd^c q^\epsilon)_{x_0}^n < f(x_0)\beta^n.$$

Since f is lower semi-continuous on $\bar{\Omega}$, there exists $r > 0$ such that

$$(dd^c q^\epsilon)_x^n \leq f(x)\beta^n ; x \in B(x_0, r).$$

Then $(dd^c q^\epsilon)^n \leq f\beta^n \leq (dd^c u)^n$ in $B(x_0, r)$ and $q^\epsilon = q + \epsilon \frac{r^2}{2} \geq q \geq u$ on $\partial B(x_0, r)$, hence $q^\epsilon \geq u$ on $B(x_0, r)$ by the comparison principle. But $q^\epsilon(x_0) = q(x_0) - \epsilon \frac{r^2}{2} = u(x_0) - \epsilon \frac{r^2}{2} < u(x_0)$ contradiction.

Hence, from the first part of the proof, we get $\Delta_H q(x_0) \geq f^{1/n}(x_0)$ for every point $x_0 \in \Omega$ and every \mathcal{C}^2 -function q in a neighborhood of x_0 such that $u \leq q$ in this neighborhood and $u(x_0) = q(x_0)$.

Assume that $f > 0$ and $f \in \mathcal{C}^\infty(\bar{\Omega})$, then there exists $g \in \mathcal{C}^\infty(\bar{\Omega})$ such that $\Delta_H g = f^{1/n}$. Hence $\varphi = u - g$ is Δ_H -subharmonic (by Proposition 3.2.10', [H94]), from which it follows $\Delta_H \varphi \geq 0$ and $\Delta_H u \geq f^{1/n}$.

In case $f > 0$ is merely continuous, we observe that

$$f = \sup\{w; w \in \mathcal{C}^\infty, f \geq w > 0\},$$

then $(dd^c u)^n \geq f\beta^n \geq w\beta^n$. Since $w > 0$ is smooth, we have $\Delta_H u \geq w^{1/n}$. Therefore, we get $\Delta_H u \geq f^{1/n}$.

In the general case $0 \leq f \in \mathcal{C}(\bar{\Omega})$, we observe that $u^\epsilon(z) = u(z) + \epsilon|z|^2$ satisfies

$$(dd^c u^\epsilon)^n \geq (f + \epsilon^n)\beta^n,$$

then

$$\Delta_H u^\epsilon \geq (f + \epsilon^n)^{1/n}.$$

Letting ϵ converges to 0, we get

$$\Delta_H u \geq f^{1/n} \text{ for all } H \in H_n^+ \text{ and } \det H = n^{-n}.$$

□

As a consequence of Proposition 2.6, we give a new description of the classical Perron-Bremermann family of subsolutions to the Dirichlet problem $Dir(\Omega, \varphi, f)$.

Definition 2.7. We denote $\mathcal{V}(\Omega, \varphi, f)$ the family of subsolutions of $Dir(\Omega, \varphi, f)$, that is $\mathcal{V}(\Omega, \varphi, f) = \{v \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega}), v|_{\partial\Omega} \leq \varphi \text{ and } \Delta_H v \geq f^{1/n} \forall H \in H_n^+, \det H = n^{-n}\}$.

Remark 2.8. We observe that $\mathcal{V}(\Omega, \varphi, f) \neq \emptyset$. Indeed, let ρ as in Definition 2.1 and $A, B > 0$ big enough, then $A\rho - B \in \mathcal{V}(\Omega, \varphi, f)$.

Furthermore, the family $\mathcal{V}(\Omega, \varphi, f)$ is stable under finite maximum, that is if $u, v \in \mathcal{V}(\Omega, \varphi, f)$ then $\max(u, v) \in \mathcal{V}(\Omega, \varphi, f)$.

3. THE PERRON-BREMERMAN ENVELOPE

Bedford and Taylor proved in [BT76] that the unique solution to $Dir(\Omega, \varphi, f)$ in a bounded strongly pseudoconvex domain with smooth boundary, is given as the envelope of Perron-Bremermann

$$u = \sup\{v; v \in \mathcal{B}(\Omega, \varphi, f)\}$$

where $\mathcal{B}(\Omega, \varphi, f) = \{v \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega}), v|_{\partial\Omega} \leq \varphi \text{ and } (dd^c v)^n \geq f\beta_n\}$.

Thanks to Proposition 2.6, we get the following corollary

Corollary 3.1. The two families $\mathcal{V}(\Omega, \varphi, f)$ and $\mathcal{B}(\Omega, \varphi, f)$ coincide, that is $\mathcal{V}(\Omega, \varphi, f) = \mathcal{B}(\Omega, \varphi, f)$.

Here we will first give an alternative description of the Perron-Bremermann envelope in a bounded SHL domain.

More precisely, we consider the upper envelope

$$\mathbb{U}(z) = \sup\{v(z); v \in \mathcal{V}(\Omega, \varphi, f)\}.$$

3.1. Continuity of the upper envelope. Following the same argument in [Wal69, Bl96], we prove the continuity of the upper envelope.

Theorem 3.2. Let $\Omega \subset \mathbb{C}^n$ be a bounded SHL domain, $0 \leq f \in \mathcal{C}(\bar{\Omega})$ and $\varphi \in \mathcal{C}(\partial\Omega)$. Then the upper envelope

$$\mathbb{U} = \sup\{v; v \in \mathcal{V}(\Omega, \varphi, f)\}$$

belongs to $\mathcal{V}(\Omega, \varphi, f)$ and $\mathbb{U} = \varphi$ on $\partial\Omega$.

Proof. Let $g \in \mathcal{C}^2(\bar{\Omega})$ be an approximation of φ such that $|g - \varphi| < \epsilon$ on $\partial\Omega$, for $\epsilon > 0$. Let also ρ the defining function as in Definition 2.1 and $A > 0$ large enough such that $v_0 := A\rho + g - \epsilon$ belongs to $\mathcal{V}(\Omega, \varphi, f)$.

Thus $v_0 \leq \mathbb{U} \leq h$, where h be the harmonic extension of φ to Ω . Then it follows that $\varphi - 2\epsilon \leq g - \epsilon \leq \mathbb{U} \leq \varphi$ on $\partial\Omega$, as ϵ tends to 0, we see that $\mathbb{U} = \varphi$ on $\partial\Omega$.

We will prove that \mathbb{U} is continuous on Ω . Fix $\epsilon > 0$ and z_0 in a compact set $K \subset \Omega$. Thanks to the continuity of h and v_0 on $\bar{\Omega}$, one can find $\delta > 0$ such that for any $z_1, z_2 \in \bar{\Omega}$ we have

$$|h(z_1) - h(z_2)| < \epsilon, |v_0(z_1) - v_0(z_2)| < \epsilon \text{ if } |z_1 - z_2| < \delta.$$

Let $a \in \mathbb{C}^n$ such that $|a| < \min(\delta, \text{dist}(K, \partial\Omega))$. Since \mathbb{U} is the upper envelope, we can find $v \in \mathcal{V}(\Omega, \varphi, f)$ such that $v(z_0 + a) > \mathbb{U}(z_0 + a) - \epsilon$ and we can assume that $v_0 \leq v$. Hence for all $z \in \bar{\Omega}$ and $w \in \partial\Omega$ we get

$$-3\epsilon < v_0(z) - \varphi(w) < v(z) - \varphi(w) < h(z) - \varphi(w) < \epsilon,$$

this implies that

$$(3.1) \quad |v(z) - \varphi(w)| < 3\epsilon \text{ if } |z - w| < \delta.$$

Then for $z \in \Omega$ and $z + a \in \partial\Omega$, we have

$$v(z + a) \leq \varphi(z + a) < v(z) + 3\epsilon.$$

We define the following function

$$v_1(z) = \begin{cases} v(z) & ; z + a \notin \bar{\Omega} \\ \max(v(z), v(z + a) - 3\epsilon) & ; z + a \in \bar{\Omega} \end{cases}$$

which is well defined, plurisubharmonic on Ω , continuous on $\bar{\Omega}$ and $v_1 \leq \varphi$ on $\partial\Omega$. Indeed, if $z \in \partial\Omega$, $z + a \notin \bar{\Omega}$ then $v_1(z) = v(z) \leq \varphi(z)$. On the other hand, if $z \in \partial\Omega$ and $z + a \in \bar{\Omega}$ then we have, from 3.1, that

$$v(z + a) - 3\epsilon < \varphi(z),$$

so $v_1(z) = \max(v(z), v(z + a) - 3\epsilon) \leq \varphi(z)$.

Moreover, we note that $\Delta_H(v(\cdot + a)) \geq f^{1/n}(\cdot + a)$, hence it follows that

$$\Delta_H v_1 \geq \min(f^{1/n}, f^{1/n}(\cdot + a)).$$

Let ω be the modulus of continuity of $f^{1/n}$ and define

$$v_2 = v_1 + \omega(|a|)(v_0 - \|v_0\|_{L^\infty(\bar{\Omega})}).$$

We claim that $v_2 \in \mathcal{V}(\Omega, \varphi, f)$. It is clear that $v_2 \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega})$ and $v_2 \leq \varphi$ on $\partial\Omega$. Moreover, One can point out that

$$\Delta_H v_2 = \Delta_H v_1 + \omega(|a|)\Delta_H v_0 \geq f^{1/n}.$$

In fact, if $\Delta_H v_1 = f^{1/n}(\cdot + a)$, by suitable choice of A we get

$$\Delta_H v_2 = f^{1/n}(\cdot + a) + \omega(|a|)\Delta_H v_0 \geq -\omega(|a|) + \omega(|a|)\Delta_H v_0 + f^{1/n} \geq f^{1/n}.$$

Hence we obtain that

$$\begin{aligned} \mathbb{U}(z_0) &\geq v_1(z_0) + \omega(|a|)v_0(z_0) - \omega(|a|)\|v_0\| \\ &\geq v(z_0 + a) - 5\epsilon \quad (\text{where } \omega(|a|) < \frac{\epsilon}{\|v_0\|}) \\ &> \mathbb{U}(z_0 + a) - 6\epsilon. \end{aligned}$$

Since $|a|$ is small and the last inequality is true for every $z_0 \in K$, then \mathbb{U} is continuous on Ω .

As the family $\mathcal{V}(\Omega, \varphi, f)$ is stable under the operation maximum, we can find a sequence $u_j \in \mathcal{V}(\Omega, \varphi, f)$ such that u_j increases almost everywhere to \mathbb{U} , then $u_j \rightarrow \mathbb{U}$ in $L^1(\Omega)$. Hence $\Delta_H \mathbb{U} = \lim \Delta_H u_j \geq f^{1/n}$ for all $H \in H_n^+$, $\det H = n^{-n}$, this implies $\mathbb{U} \in \mathcal{V}(\Omega, \varphi, f)$. \square

Proposition 3.3. *Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly hyperconvex Lipschitz domain, $0 \leq f \in \mathcal{C}(\bar{\Omega})$ and $\varphi \in \mathcal{C}(\partial\Omega)$. Then the Dirichlet problem $\text{Dir}(\Omega, \varphi, f)$ has a unique solution \mathbf{U} . Moreover the solution is given by*

$$\mathbf{U} = \sup\{v; v \in \mathcal{V}(\Omega, \varphi, f)\}$$

where

$$\mathcal{V} = \{v \in \text{PSH}(\Omega) \cap \mathcal{C}(\bar{\Omega}), v|_{\partial\Omega} \leq \varphi \text{ and } \Delta_H v \geq f^{1/n} \forall H \in H_n^+, \det H = n^{-n}\}$$

and Δ_H be the laplacian associated to a positive definite Hermitian matrix H as in (2.1).

Proof. The uniqueness follows from the comparison principle ([BT76]). On the other hand, Theorem 3.2 implies that our domain Ω is B -regular in the sense of Sibony ([Sib87]). Therefore existence and uniqueness of the solution follows from Theorem 4.1 in [Bl96]. The description of the solution given in the proposition follows from Corollary 3.1 and Theorem 3.2. \square

Remark 3.4. *Let $\varphi_1, \varphi_2 \in \mathcal{C}(\partial\Omega)$ and $f_1, f_2 \in \mathcal{C}(\bar{\Omega})$, then the solutions $\mathbf{U}_1 = \mathbf{U}(\Omega, \varphi_1, f_1)$, $\mathbf{U}_2 = \mathbf{U}(\Omega, \varphi_2, f_2)$ satisfy the following stability estimate*

$$(3.2) \quad \|\mathbf{U}_1 - \mathbf{U}_2\|_{L^\infty(\bar{\Omega})} \leq d^2 \|f_1 - f_2\|_{L^\infty(\bar{\Omega})}^{1/n} + \|\varphi_1 - \varphi_2\|_{L^\infty(\partial\Omega)}$$

where $d := \text{diam}(\Omega)$. Indeed, fix $z_0 \in \Omega$ and define

$$v_1(z) = \|f_1 - f_2\|_{L^\infty(\bar{\Omega})}^{1/n} (|z - z_0|^2 - d^2) + \mathbf{U}_2(z)$$

and

$$v_2(z) = \mathbf{U}_1(z) + \|\varphi_1 - \varphi_2\|_{L^\infty(\partial\Omega)}.$$

It is clear that $v_1, v_2 \in \text{PSH}(\Omega) \cap \mathcal{C}(\bar{\Omega})$. Hence, by the comparison principle, we get $v_1 \leq v_2$ on $\bar{\Omega}$. Then we conclude that

$$\mathbf{U}_2 - \mathbf{U}_1 \leq d^2 \|f_1 - f_2\|_{L^\infty(\bar{\Omega})}^{1/n} + \|\varphi_1 - \varphi_2\|_{L^\infty(\partial\Omega)}$$

Reversing the roles of \mathbf{U}_1 and \mathbf{U}_2 , we get the inequality (3.2).

We will need in Section 5 an estimate, proved by Blocki in [Bl93], for the $L^n - L^1$ stability of solutions to the Dirichlet problem $\text{Dir}(\Omega, \varphi, f)$

$$(3.3) \quad \|\mathbf{U}_1 - \mathbf{U}_2\|_{L^n(\Omega)} \leq \lambda(\Omega) \|\varphi_1 - \varphi_2\|_{L^\infty(\partial\Omega)} + \frac{r^2}{4} \|f_1 - f_2\|_{L^1(\Omega)}^{1/n}$$

where $r = \min\{r' > 0 : \Omega \subset B(z_0, r') \text{ for some } z_0 \in \mathbb{C}^n\}$.

4. THE MODULUS OF CONTINUITY OF PERRON-BREMERMANN ENVELOPE

Recall that a real function ω on $[0, l]$, $0 < l < \infty$, is called a modulus of continuity if ω is continuous, subadditive, nondecreasing and $\omega(0) = 0$.

In general, ω fails to be concave, we denote $\bar{\omega}$ to be the minimal concave majorant of ω . The following property of the minimal concave majorant $\bar{\omega}$ is well known (see [Kor82] and [Ch14]).

Lemma 4.1. *Let ω be a modulus of continuity on $[0, l]$ and $\bar{\omega}$ be the minimal concave majorant of ω . Then $\omega(\eta t) < \bar{\omega}(\eta t) < (1 + \eta)\omega(t)$ for any $t > 0$ and $\eta > 0$.*

4.1. Modulus of continuity of the solution. Now, we will start the first step to establish an estimate for the modulus of continuity of the solution to $Dir(\Omega, \varphi, f)$. For this reason, it is natural to require the relation between the modulus of continuity of \mathbf{U} and the modulus of continuity of sub-barrier and super-barrier. Thus, we present the following proposition

Proposition 4.2. *Let $\Omega \subset \mathbb{C}^n$ be a bounded SHL domain, $\varphi \in \mathcal{C}(\partial\Omega)$ and $0 \leq f \in \mathcal{C}(\bar{\Omega})$. Suppose that there exist $v \in \mathcal{V}(\Omega, \varphi, f)$ and $w \in SH(\Omega) \cap \mathcal{C}(\bar{\Omega})$ such that $v = \varphi = -w$ on $\partial\Omega$, then there is a constant $C > 0$ depends on $diam(\Omega)$ such that the modulus of continuity of \mathbf{U} satisfies*

$$\omega_{\mathbf{U}}(t) \leq C \max\{\omega_v(t), \omega_w(t), \omega_{f^{1/n}}(t)\}.$$

Proof. Let us put $g(t) := \max(\omega_v(t), \omega_w(t), \omega_{f^{1/n}}(t))$ and $d := diam(\Omega)$. As $v = \varphi = -w$ on $\partial\Omega$ we have for all $z \in \bar{\Omega}$ and $\xi \in \partial\Omega$

$$-g(|z - \xi|) \leq v(z) - \varphi(\xi) \leq \mathbf{U}(z) - \varphi(\xi) \leq -w(z) - \varphi(\xi) \leq g(|z - \xi|).$$

Hence we get

$$(4.1) \quad |\mathbf{U}(z) - \mathbf{U}(\xi)| \leq g(|z - \xi|); \forall z \in \bar{\Omega}, \forall \xi \in \partial\Omega.$$

Fix a point $z_0 \in \Omega$, for any small vector $\tau \in \mathbb{C}^n$, we set $\Omega_{-\tau} := \{z - \tau; z \in \Omega\}$ and define in $\Omega \cap \Omega_{-\tau}$ the function

$$v_1(z) = \mathbf{U}(z + \tau) + g(|\tau|)|z - z_0|^2 - d^2 g(|\tau|) - g(|\tau|)$$

which is well defined psh function in $\Omega \cap \Omega_{-\tau}$ and continuous on $\bar{\Omega} \cap \bar{\Omega}_{-\tau}$. By (4.1), if $z \in \bar{\Omega} \cap \partial\Omega_{-\tau}$ we can see that

$$(4.2) \quad v_1(z) - \mathbf{U}(z) \leq g(|\tau|) + g(|\tau|)|z - z_0|^2 - d^2 g(|\tau|) - g(|\tau|) \leq 0.$$

Moreover, we assert that $\Delta_H v_1 \geq f^{1/n}$ in $\Omega \cap \Omega_{-\tau}$ for all $H \in H_n^+$, $\det H = n^{-n}$. Indeed, we have

$$\begin{aligned} \Delta_H v_1(z) &\geq f^{1/n}(z + \tau) + g(|\tau|)\Delta_H(|z - z_0|^2) \\ &\geq f^{1/n}(z + \tau) + g(|\tau|) \\ &\geq f^{1/n}(z + \tau) + |f^{1/n}(z + \tau) - f^{1/n}(z)| \\ &\geq f^{1/n}(z) \end{aligned}$$

for all $H \in H_n^+$ and $\det H = n^{-n}$.

Hence, by the last properties of v_1 , we find that

$$V_\tau(z) = \begin{cases} \mathbf{U}(z) & ; z \in \bar{\Omega} \setminus \Omega_{-\tau} \\ \max(\mathbf{U}(z), v_1(z)) & ; z \in \bar{\Omega} \cap \Omega_{-\tau} \end{cases}$$

is well defined function and belongs to $PSH(\Omega) \cap \mathcal{C}(\bar{\Omega})$. It is clear that $\Delta_H V_\tau \geq f^{1/n}$ for all $H \in H_n^+$, $\det H = n^{-n}$. We claim that $V_\tau = \varphi$ on $\partial\Omega$. If $z \in \partial\Omega \setminus \Omega_{-\tau}$ then $V_\tau(z) = \mathbf{U}(z) = \varphi(z)$. On the other hand $z \in \partial\Omega \cap \Omega_{-\tau}$, by (4.2) we get $V_\tau(z) = \max(\mathbf{U}(z), v_1(z)) = \mathbf{U}(z) = \varphi(z)$. Consequently $V_\tau \in \mathcal{V}(\Omega, \varphi, f)$ and this implies that

$$V_\tau(z) \leq \mathbf{U}(z); \forall z \in \bar{\Omega}.$$

Then we have for all $z \in \bar{\Omega} \cap \Omega_{-\tau}$

$$\mathbb{U}(z + \tau) + g(|\tau|)|z - z_0|^2 - d^2 g(|\tau|) - g(|\tau|) \leq \mathbb{U}(z).$$

Hence,

$$\mathbb{U}(z + \tau) - \mathbb{U}(z) \leq (d^2 + 1)g(|\tau|) - g(|\tau|)|z - z_0|^2 \leq Cg(|\tau|).$$

Reversing the roles of $z + \tau$ and z , we get

$$|\mathbb{U}(z + \tau) - \mathbb{U}(z)| \leq Cg(|\tau|); \forall z, z + \tau \in \bar{\Omega}.$$

Thus, we finally get

$$\omega_{\mathbb{U}}(|\tau|) \leq C \max(\omega_v(|\tau|), \omega_w(|\tau|), \omega_{f^{1/n}}(|\tau|)).$$

□

Remark 4.3. Let H_φ be the harmonic extension of φ in a bounded SHL domain Ω , we can replace w in the last proposition by H_φ . It is known in the classical harmonic analysis (see [Ai10]) that the harmonic extension H_φ has not, in general, the same modulus of continuity of φ .

Let us define, for small positive t , the modulus of continuity

$$\psi_{\alpha,\beta}(t) = (-\log(t))^{-\alpha} t^\beta$$

with $\alpha \geq 0$ and $0 \leq \beta < 1$. It is clear that $\psi_{\alpha,0}$ is weaker than the Hölder continuity and $\psi_{0,\beta}$ is the Hölder continuity. It was shown in [Ai02] that $\omega_{H_\varphi}(t) \leq c\psi_{0,\beta}(t)$ for some $c > 0$ if $\omega_\varphi(t) \leq c_1\psi_{0,\beta}(t)$ for $\beta < \beta_0$ where $\beta_0 < 1$ depending only on n and the Lipschitz constant of the defining function ρ . Moreover, a similar result was proved in [Ai10] for the modulus of continuity $\psi_{\alpha,0}(t)$. However, the same argument of Aikawa gives that $\omega_{H_\varphi}(t) \leq c\psi_{\alpha,\beta}(t)$ for some $c > 0$ if $\omega_\varphi(t) \leq c_1\psi_{\alpha,\beta}(t)$ for $\alpha \geq 0$ and $0 \leq \beta < \beta_0 < 1$.

Hence, this leads us to the conclusion that if there exists a barrier v to the Dirichlet problem such that $v = \varphi$ on $\partial\Omega$ and $\omega_v(t) \leq \lambda\psi_{\alpha,\beta}(t)$ with α, β as above, then the last proposition gives

$$\omega_{\mathbb{U}} \leq \lambda_1 \max\{\psi_{\alpha,\beta}(t), \omega_{f^{1/n}}(t)\},$$

where $\lambda_1 > 0$ depending on λ and $\text{diam}(\Omega)$.

4.2. Construction of barriers. In this subsection, we will construct a subsolution to Dirichlet problem with the boundary value φ and estimate its modulus of continuity.

Proposition 4.4. Let $\Omega \subset \mathbb{C}^n$ be a bounded SHL domain, assume that $\varphi \in \mathcal{C}(\partial\Omega)$ and $0 \leq f \in \mathcal{C}(\bar{\Omega})$. Then there exists a subsolution $v \in \mathcal{V}(\Omega, \varphi, f)$ such that $v = \varphi$ on $\partial\Omega$ and the modulus of continuity of v satisfies the following inequality

$$\omega_v(t) \leq \lambda(1 + \|f\|_{L^\infty(\bar{\Omega})}^{1/n}) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}$$

where $\lambda > 0$ depends on Ω .

Observe that we do not assume any smoothness on $\partial\Omega$.

Proof. First of all, let us fix $\xi \in \partial\Omega$, we claim that there exists $v_\xi \in \mathcal{V}(\Omega, \varphi, f)$ such that $v_\xi(\xi) = \varphi(\xi)$. It is sufficient to prove that there exists a constant $C > 0$ depending on Ω such that for every point $\xi \in \partial\Omega$ and $\varphi \in \mathcal{C}(\partial\Omega)$, there is a function $h_\xi \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega})$ such that

- 1) $h_\xi(z) \leq \varphi(z), \forall z \in \partial\Omega$
- 2) $h_\xi(\xi) = \varphi(\xi)$
- 3) $\omega_{h_\xi}(t) \leq C\omega_\varphi(t^{1/2})$.

Assume this is true, we fix $z_0 \in \Omega$ and choose $K_1 := \sup_{\bar{\Omega}} f^{1/n} \geq 0$, hence

$$\Delta_H(K_1|z - z_0|^2) = K_1\Delta_H|z - z_0|^2 \geq f^{1/n}, \quad \forall H \in H_n^+, \det H = n^{-n},$$

we also put $K_2 = K_1|\xi - z_0|^2$. Then for the continuous function

$$\tilde{\varphi}(z) := \varphi(z) - K_1|z - z_0|^2 + K_2,$$

we have h_ξ such that 1), 2) and 3) hold.

Then the desired function $v_\xi \in \mathcal{V}(\Omega, \varphi, f)$ is given by

$$v_\xi(z) = h_\xi(z) + K_1|z - z_0|^2 - K_2$$

Because, $h_\xi(z) \leq \tilde{\varphi}(z) = \varphi(z) - K_1|z - z_0|^2 + K_2$ on $\partial\Omega$, so $v_\xi(z) \leq \varphi$ on $\partial\Omega$ and $v_\xi(\xi) = \varphi(\xi)$.

Moreover, it is clear that

$$\Delta_H v_\xi = \Delta_H h_\xi + K_1\Delta_H(|z - z_0|^2) \geq f^{1/n}, \quad \forall H \in H_n^+, \det H = n^{-n}.$$

Furthermore, using the hypothesis of h_ξ , we can control the modulus of continuity of v_ξ

$$\begin{aligned} \omega_{v_\xi}(t) &= \sup_{|z-y| \leq t} |v_\xi(z) - v_\xi(y)| \leq \omega_{h_\xi}(t) + K_1\omega_{|z-z_0|^2}(t) \\ &\leq C\omega_{\tilde{\varphi}}(t^{1/2}) + 4d^{3/2}K_1t^{1/2} \\ &\leq C\omega_\varphi(t^{1/2}) + 2dK_1(C + 2d^{1/2})t^{1/2} \\ &\leq (C + 2d^{1/2})(1 + 2dK_1) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}. \end{aligned}$$

Hence, we conclude that

$$\omega_{v_\xi}(t) \leq \lambda(1 + K_1) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}$$

where $\lambda := (C + 2d^{1/2})(1 + 2d)$ is a positive constant depending on Ω .

Now we will construct $h_\xi \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega})$ which satisfies the three conditions above.

Let $B > 0$ large enough such that the function

$$g(z) = B\rho(z) - |z - \xi|^2$$

is psh in Ω . Let $\bar{\omega}_\varphi$ be the minimal concave majorant of ω_φ and define

$$\chi(x) = -\bar{\omega}_\varphi((-x)^{1/2})$$

which is convex nondecreasing function on $[-d^2, 0]$. Now fix $r > 0$ so small that $|g(z)| \leq d^2$ in $B(\xi, r) \cap \Omega$ and define for $z \in B(\xi, r) \cap \Omega$ the function

$$h(z) = \chi \circ g(z) + \varphi(\xi).$$

It is clear that h is continuous psh function on $B(\xi, r) \cap \Omega$ and we see that $h(z) \leq \varphi(z)$ if $z \in B(\xi, r) \cap \partial\Omega$ and $h(\xi) = \varphi(\xi)$. Moreover by the subadditivity of $\bar{\omega}_\varphi$ and Lemma 4.1 we have

$$\begin{aligned} \omega_h(t) &= \sup_{|z-y| \leq t} |h(z) - h(y)| \\ &\leq \sup_{|z-y| \leq t} \bar{\omega}_\varphi \left[\left| |z - \xi|^2 - |y - \xi|^2 - B(\rho(z) - \rho(y)) \right|^{1/2} \right] \\ &\leq \sup_{|z-y| \leq t} \bar{\omega}_\varphi \left[(|z - y|(2d + B_1))^{1/2} \right] \\ &\leq C \cdot \omega_\varphi(t^{1/2}) \end{aligned}$$

where $C := 1 + (2d + B_1)^{1/2}$ depends on Ω .

Recall that $\xi \in \partial\Omega$ and fix $0 < r_1 < r$ and $\gamma_1 \geq d/r_1$ such that

$$-\gamma_1 \bar{\omega}_\varphi \left[(|z - \xi|^2 - B\rho(z))^{1/2} \right] \leq \inf_{\partial\Omega} \varphi - \sup_{\partial\Omega} \varphi,$$

for $z \in \partial\Omega \cap \partial B(\xi, r_1)$. Set $\gamma_2 = \inf_{\partial\Omega} \varphi$, then it follows that

$$\gamma_1(h(z) - \varphi(\xi)) + \varphi(\xi) \leq \gamma_2 \text{ for } z \in \partial B(\xi, r_1) \cap \bar{\Omega}.$$

Now let us put

$$h_\xi(z) = \begin{cases} \max[\gamma_1(h(z) - \varphi(\xi)) + \varphi(\xi), \gamma_2] & ; z \in \bar{\Omega} \cap B(\xi, r_1) \\ \gamma_2 & ; z \in \bar{\Omega} \setminus B(\xi, r_1) \end{cases}$$

which is well defined plurisubharmonic function on Ω , continuous on $\bar{\Omega}$ and satisfies that $h_\xi(z) \leq \varphi(z)$ for all $z \in \partial\Omega$. Indeed, on $\partial\Omega \cap B(\xi, r_1)$ we have

$$\gamma_1(h(z) - \varphi(\xi)) + \varphi(\xi) = -\gamma_1 \bar{\omega}_\varphi(|z - \xi|) + \varphi(\xi) \leq -\bar{\omega}_\varphi(|z - \xi|) + \varphi(\xi) \leq \varphi(z).$$

Hence it is clear that h_ξ satisfies the three conditions above.

We have just proved that for each $\xi \in \partial\Omega$, there is a function

$$v_\xi \in \mathcal{V}(\Omega, \varphi, f), \quad v_\xi(\xi) = \varphi(\xi) \text{ and } \omega_{v_\xi}(t) \leq \lambda(1 + K_1) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}.$$

Let us set

$$v(z) = \sup \{v_\xi(z); \xi \in \partial\Omega\}.$$

We can note $0 \leq \omega_v(t) \leq \lambda(1 + K_1) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}$, then $\omega_v(t)$ converges to zero when t converges to zero. Consequently, we get $v \in \mathcal{C}(\bar{\Omega})$ and $v = v^* \in PSH(\Omega)$. Thanks to Choquet lemma, we can choose a nondecreasing sequence (v_j) , where $v_j \in \mathcal{V}(\Omega, \varphi, f)$, converging to v almost everywhere. This implies that

$$\Delta_H v = \lim_{j \rightarrow \infty} \Delta_H v_j \geq f^{1/n}, \forall H \in H_n^+, \det H = n^{-n}.$$

It is clear that $v(\xi) = \varphi(\xi)$ for any $\xi \in \partial\Omega$. Finally, we get $v \in \mathcal{V}(\Omega, \varphi, f)$, $v = \varphi$ on $\partial\Omega$ and $\omega_v(t) \leq \lambda(1 + K_1) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}$. \square

Remark 4.5. *If we assume that Ω has a smooth boundary and φ is $C^{1,1}$ -smooth, then it is possible to construct a Lipschitz barrier v to the Dirichlet problem $\text{Dir}(\Omega, \varphi, f)$ (see Theorem 6.2 in [BT76]).*

Corollary 4.6. *Under the same assumption of Proposition 4.4. There exists a plurisuperharmonic function $\tilde{v} \in \mathcal{C}(\bar{\Omega})$ such that $\tilde{v} = \varphi$ on $\partial\Omega$ and*

$$\omega_{\tilde{v}}(t) \leq \lambda(1 + \|f\|_{L^\infty(\bar{\Omega})}^{1/n}) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\},$$

where $\lambda > 0$ depends on Ω .

Proof. We can do the same construction as in the proof of Proposition 4.4 for the function $\varphi_1 = -\varphi \in \mathcal{C}(\partial\Omega)$, then we get $v_1 \in \mathcal{V}(\Omega, \varphi_1, f)$ such that $v_1 = \varphi_1$ on $\partial\Omega$ and $\omega_{v_1}(t) \leq (1 + \|f\|_{L^\infty(\bar{\Omega})}^{1/n}) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}$. Hence, we set $\tilde{v} = -v_1$ which is a plurisuperharmonic function on Ω , continuous on $\bar{\Omega}$ and satisfies $\tilde{v} = \varphi$ on $\partial\Omega$ and $\omega_{\tilde{v}}(t) \leq \lambda(1 + \|f\|_{L^\infty(\bar{\Omega})}^{1/n}) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}$. \square

4.3. Proof of Theorem A. Thanks to Proposition 4.4, we obtain a subsolution $v \in \mathcal{V}(\Omega, \varphi, f)$, $v = \varphi$ on $\partial\Omega$ and

$$\omega_v(t) \leq \lambda(1 + \|f\|_{L^\infty(\bar{\Omega})}^{1/n}) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}.$$

Observing Corollary 4.6, we get $w \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega})$ such that $w = -\varphi$ on $\partial\Omega$ and

$$\omega_w(t) \leq \lambda(1 + \|f\|_{L^\infty(\bar{\Omega})}^{1/n}) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}$$

where $\lambda > 0$ constant. Applying the Proposition 4.2 we get the wanted result, that is

$$\omega_{\mathbb{U}}(t) \leq \eta(1 + \|f\|_{L^\infty(\bar{\Omega})}^{1/n}) \max\{\omega_\varphi(t^{1/2}), \omega_{f^{1/n}}(t), t^{1/2}\}$$

where $\eta > 0$ depends on Ω .

Corollary 4.7. *Let Ω be a bounded SHL domain in \mathbb{C}^n . Let $\varphi \in \mathcal{C}^{0,\alpha}(\partial\Omega)$ and $0 \leq f^{1/n} \in \mathcal{C}^{0,\beta}(\bar{\Omega})$, $0 < \alpha, \beta \leq 1$. Then the solution \mathbb{U} to the Dirichlet problem $\text{Dir}(\Omega, \varphi, f)$ belongs to $\mathcal{C}^{0,\gamma}(\bar{\Omega})$ for $\gamma = \min(\beta, \alpha/2)$.*

The following example illustrates that the estimate of $\omega_{\mathbb{U}}$ in Theorem A is optimal.

Example 4.8. *Let ψ be a concave modulus of continuity on $[0, 1]$ and*

$$\varphi(z) = -\psi[\sqrt{(1 + \text{Re}z_1)/2}], \text{ for } z = (z_1, z_2, \dots, z_n) \in \partial\mathbb{B} \subset \mathbb{C}^n.$$

It is easy to show that $\varphi \in \mathcal{C}(\partial\mathbb{B})$ with modulus of continuity

$$\omega_\varphi(t) \leq C\psi(t)$$

for some $C > 0$.

Let $v(z) = -(1 + \text{Re}z_1)/2 \in PSH(\mathbb{B}) \cap \mathcal{C}(\bar{\mathbb{B}})$ and $\chi(\lambda) = -\psi(\sqrt{-\lambda})$ is convex increasing function on $[-1, 0]$. Hence we get that

$$u(z) = \chi \circ v(z) \in PSH(\mathbb{B}) \cap \mathcal{C}(\bar{\mathbb{B}})$$

and satisfies $(dd^c u)^n = 0$ in \mathbb{B} and $u = \varphi$ on $\partial\mathbb{B}$. The modulus of continuity of \mathbb{U} , $\omega_{\mathbb{U}}(t)$, has the estimate

$$C_1\psi(t^{1/2}) \leq \omega_{\mathbb{U}}(t) \leq C_2\psi(t^{1/2})$$

for $C_1, C_2 > 0$.

Indeed, let $z_0 = (-1, 0, \dots, 0)$ and $z = (z_1, 0, \dots, 0) \in \mathbb{B}$ where $z_1 = -1 + 2t$ and $0 \leq t \leq 1$. Hence, by Lemma 4.1, we see that

$$\psi(t^{1/2}) = \psi[\sqrt{|z - z_0|/2}] = \psi[\sqrt{(1 + \operatorname{Re} z_1)/2}] = |\mathbf{U}(z) - \mathbf{U}(z_0)| \leq \omega_{\mathbf{U}}(2t) \leq 3\omega_{\mathbf{U}}(t).$$

Definition 4.9. Let ψ be a modulus of continuity, $E \subset \mathbb{C}^n$ be a bounded set and $g \in \mathcal{C} \cap L^\infty(E)$. We define the norm of g with respect to ψ (ψ -norm) as follows

$$\|g\|_\psi := \sup_{z \in E} |g(z)| + \sup_{z \neq y \in E} \frac{|g(z) - g(y)|}{\psi(|z - y|)}$$

Proposition 4.10. Let $\Omega \subset \mathbb{C}^n$ be a bounded SHL domain, $\varphi \in \mathcal{C}(\partial\Omega)$ with modulus of continuity ψ_1 and $f^{1/n} \in \mathcal{C}(\bar{\Omega})$ with modulus of continuity ψ_2 . Then there exists a constant $C > 0$ depending on Ω such that

$$\|\mathbf{U}\|_\psi \leq C(1 + \|f\|_{L^\infty(\bar{\Omega})}^{1/n}) \max\{\|\varphi\|_{\psi_1}, \|f^{1/n}\|_{\psi_2}\}$$

where $\psi(t) = \max\{\psi_1(t^{1/2}), \psi_2(t)\}$.

Proof. By hypothesis, we see that $\|\varphi\|_{\psi_1} < \infty$ and $\|f^{1/n}\|_{\psi_2} < \infty$. Let $z \neq y \in \bar{\Omega}$, by Theorem A, we get

$$\begin{aligned} |\mathbf{U}(z) - \mathbf{U}(y)| &\leq \eta(1 + \|f\|_{L^\infty(\bar{\Omega})}^{1/n}) \max\{\omega_\varphi(|z - y|^{1/2}), \omega_{f^{1/n}}(|z - y|)\} \\ &\leq \eta(1 + \|f\|_{L^\infty(\bar{\Omega})}^{1/n}) \max\{\|\varphi\|_{\psi_1} \psi_1(|z - y|^{1/2}), \|f^{1/n}\|_{\psi_2} \psi_2(|z - y|)\} \\ &\leq \eta(1 + \|f\|_{L^\infty(\bar{\Omega})}^{1/n}) \max\{\|\varphi\|_{\psi_1}, \|f^{1/n}\|_{\psi_2}\} \psi(|z - y|) \end{aligned}$$

where $\psi(|z - y|) = \max\{\psi_1(|z - y|^{1/2}), \psi_2(|z - y|)\}$.

Hence we have

$$\sup_{z \neq y \in \bar{\Omega}} \frac{|\mathbf{U}(z) - \mathbf{U}(y)|}{\psi(|z - y|)} \leq \eta(1 + \|f\|_{L^\infty(\bar{\Omega})}^{1/n}) \max\{\|\varphi\|_{\psi_1}, \|f^{1/n}\|_{\psi_2}\}$$

where $\eta \geq d^2 + 1$ and $d = \operatorname{diam}(\Omega)$ (see Proposition 4.2). From Remark 3.2, we note that

$$\|\mathbf{U}\|_{L^\infty(\bar{\Omega})} \leq d^2 \|f\|_{L^\infty(\bar{\Omega})}^{1/n} + \|\varphi\|_{L^\infty(\partial\Omega)} \leq \eta \max\{\|\varphi\|_{\psi_1}, \|f^{1/n}\|_{\psi_2}\}$$

Then we can conclude that

$$\|\mathbf{U}\|_\psi \leq 2\eta(1 + \|f\|_{L^\infty(\bar{\Omega})}^{1/n}) \max\{\|\varphi\|_{\psi_1}, \|f^{1/n}\|_{\psi_2}\}.$$

□

Finally, it is natural to try to relate the modulus of continuity of $\mathbf{U} := \mathbf{U}(\Omega, \varphi, f)$ to the modulus of continuity of $\mathbf{U}_0 := \mathbf{U}(\Omega, \varphi, 0)$ the solution to Bremermann problem in a bounded SHL domain.

Proposition 4.11. Let Ω be a bounded SHL domain in \mathbb{C}^n , $f \in \mathcal{C}(\bar{\Omega})$ and $\varphi \in \mathcal{C}(\partial\Omega)$. Then there exists a positive constant $C = C(\Omega)$ such that

$$\omega_{\mathbf{U}}(t) \leq C(1 + \|f\|_{L^\infty(\bar{\Omega})}^{1/n}) \max\{\omega_{\mathbf{U}_0}(t), \omega_{f^{1/n}}(t)\}.$$

Proof. First, we search a subsolution $v \in \mathcal{V}(\Omega, \varphi, f)$ such that $v|_{\partial\Omega} = \varphi$ and estimate its modulus of continuity. Since Ω is bounded SHL domain, there exists a Lipschitz defining function ρ on $\bar{\Omega}$. Let us define the function

$$v(z) = \mathbf{U}_0(z) + A\rho(z)$$

where $A := \|f\|_{L^\infty}^{1/n}/c$ and $c > 0$ as in the Definition 2.1. It is clear that $v \in \mathcal{V}(\Omega, \varphi, f)$, $v = \varphi$ on $\partial\Omega$ and $\omega_v(t) \leq \tilde{C}\omega_{\mathbf{U}_0}(t)$ where $\tilde{C} := \gamma(1 + \|f\|_{L^\infty(\bar{\Omega})}^{1/n})$ and $\gamma \geq 1$ depends on Ω . On the other hand, by the comparison principle we get that $\mathbf{U} \leq \mathbf{U}_0$. Hence

$$v \leq \mathbf{U} \leq \mathbf{U}_0 \text{ in } \Omega \text{ and } v = \mathbf{U} = \mathbf{U}_0 = \varphi \text{ on } \partial\Omega.$$

Thanks to Proposition 4.2, there exists $\lambda > 0$ depending on Ω such that

$$\omega_{\mathbf{U}}(t) \leq \lambda \max\{\omega_v(t), \omega_{\mathbf{U}_0}(t), \omega_{f^{1/n}}(t)\}.$$

Hence, the following inequality holds for some $C > 0$ depending on Ω

$$\omega_{\mathbf{U}}(t) \leq C(1 + \|f\|_{L^\infty(\bar{\Omega})}^{1/n}) \max\{\omega_{\mathbf{U}_0}(t), \omega_{f^{1/n}}(t)\}.$$

□

5. HÖLDER CONTINUOUS SOLUTIONS FOR THE DIRICHLET PROBLEM WITH L^p DENSITY

In this section we will prove the existence and the Hölder continuity of the solution to Dirichlet problem $Dir(\Omega, \varphi, f)$ when $f \in L^p(\Omega)$, $p > 1$ in a bounded SHL domain.

It is well known in [Ko98] that there exists a weak continuous solution to this problem when Ω is a bounded strongly pseudoconvex domain with smooth boundary.

The Hölder continuity of this solution was studied in [GKZ08] under some additional conditions on the density and on the boundary data, that is when f is bounded near the boundary and $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$.

An essential method in this study is played by an a priori weak stability estimate of the solution which is still true when Ω is a bounded SHL domain. More precisely, we have the following theorem

Theorem 5.1. ([GKZ08]). *Fix $0 \leq f \in L^p(\Omega)$, $p > 1$. Let u, v be two bounded plurisubharmonic functions in Ω such that $(dd^c u)^n = f\beta^n$ in Ω and let $u \geq v$ on $\partial\Omega$. Fix $r \geq 1$ and $0 \leq \gamma < r/(nq+r)$, $1/p+1/q=1$. Then there exists a uniform constant $C = C(\gamma, n, q) > 0$ such that*

$$\sup_{\Omega}(v - u) \leq C(1 + \|f\|_{L^p(\Omega)}^\tau) \|(v - u)_+\|_{L^r(\Omega)}^\gamma$$

where $\tau := \frac{1}{n} + \frac{\gamma q}{r - \gamma(r + nq)}$ and $(v - u)_+ := \max(v - u, 0)$.

It was constructed in [GKZ08] a Lipschitz continuous barrier to the Dirichlet problem when $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$ and f is bounded near the boundary. Moreover, it was shown in this case that the total mass of $\Delta \mathbf{U}$ is finite in Ω . Finally, they conclude that $\mathbf{U} \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$ for any $\alpha < 2/(nq + 1)$. However, the following theorem summarizes the work introduced in [GKZ08]

Theorem 5.2. ([GKZ08]). *Let $0 \leq f \in L^p(\Omega)$, for some $p > 1$ and $\varphi \in \mathcal{C}(\partial\Omega)$. Suppose that there exists $v, w \in PSH(\Omega) \cap \mathcal{C}^{0,\alpha}(\bar{\Omega})$ such that $v \leq \mathbb{U} \leq -w$ on $\bar{\Omega}$ and $v = \varphi = -w$ on $\partial\Omega$. If the total mass of $\Delta\mathbb{U}$ is finite in Ω , then $\mathbb{U} \in \mathcal{C}^{0,\alpha'}(\bar{\Omega})$ for $\alpha' < \min\{\alpha, 2/(nq + 1)\}$.*

Here let $\Omega \subset \mathbb{C}^n$ be a bounded SHL domain. Using the stability theorem 5.1 we will ensure the existence of the solution to the Dirichlet problem $Dir(\Omega, \varphi, f)$.

Proposition 5.3. *Let $\Omega \subset \mathbb{C}^n$ be a bounded SHL domain, $\varphi \in \mathcal{C}(\partial\Omega)$ and $f \in L^p(\Omega)$ for some $p > 1$. Then there exists a unique solution \mathbb{U} to the Dirichlet problem $Dir(\Omega, \varphi, f)$.*

Proof. Let (f_j) be a sequence of smooth functions on $\bar{\Omega}$ which converges to f in $L^p(\Omega)$. Thanks to Proposition 3.3, there exists a unique solution \mathbb{U}_j to $Dir(\Omega, \varphi, f_j)$ that is $\mathbb{U}_j \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega})$, $\mathbb{U}_j = \varphi$ on $\partial\Omega$ and $(dd^c\mathbb{U}_j)^n = f_j\beta^n$ in Ω . We claim that

$$(5.1) \quad \|\mathbb{U}_k - \mathbb{U}_j\|_{L^\infty(\bar{\Omega})} \leq A(1 + \|f_k\|_{L^p(\Omega)}^\tau)(1 + \|f_j\|_{L^p(\Omega)}^\tau)\|f_k - f_j\|_{L^1(\Omega)}^{\gamma/n}$$

where $0 \leq \gamma < 1/(q + 1)$ fixed, $\tau := \frac{1}{n} + \frac{\gamma q}{n - \gamma n(1+q)}$ and $A = A(\gamma, n, q, \text{diam}(\Omega))$.

Indeed, by the stability theorem 5.1 and for $r = n$, we get that

$$\sup_{\Omega}(\mathbb{U}_k - \mathbb{U}_j) \leq C(1 + \|f_j\|_{L^p(\Omega)}^\tau)\|(\mathbb{U}_k - \mathbb{U}_j)_+\|_{L^n(\Omega)}^\gamma \leq C(1 + \|f_j\|_{L^p(\Omega)}^\tau)\|\mathbb{U}_k - \mathbb{U}_j\|_{L^n(\Omega)}^\gamma$$

where $0 \leq \gamma < 1/(q + 1)$ fixed and $C = C(\gamma, n, q) > 0$.

Hence by the $L^n - L^1$ stability theorem in [Bl93] (see here Remark 3.2), we get

$$\|\mathbb{U}_k - \mathbb{U}_j\|_{L^n(\Omega)} \leq \tilde{C}\|f_k - f_j\|_{L^1(\Omega)}^{1/n},$$

where \tilde{C} depends on $\text{diam}(\Omega)$.

Then, by combining the last two inequalities, we get

$$\sup_{\Omega}(\mathbb{U}_k - \mathbb{U}_j) \leq C\tilde{C}^\gamma(1 + \|f_j\|_{L^p(\Omega)}^\tau)\|f_k - f_j\|_{L^1(\Omega)}^{\gamma/n}$$

Reversing the roles of \mathbb{U}_j and \mathbb{U}_k we see that

$$\sup_{\Omega}(\mathbb{U}_j - \mathbb{U}_k) \leq C\tilde{C}^\gamma(1 + \|f_k\|_{L^p(\Omega)}^\tau)\|f_k - f_j\|_{L^1(\Omega)}^{\gamma/n}$$

Hence we conclude that

$$\|\mathbb{U}_k - \mathbb{U}_j\|_{L^\infty(\Omega)} \leq C\tilde{C}^\gamma(1 + \|f_k\|_{L^p(\Omega)}^\tau)(1 + \|f_j\|_{L^p(\Omega)}^\tau)\|f_k - f_j\|_{L^1(\Omega)}^{\gamma/n}$$

Since $\mathbb{U}_k = \mathbb{U}_j = \varphi$ on $\partial\Omega$, we get the inequality (5.1).

Since f_j converges to f in $L^p(\Omega)$, there is a uniform constant $B > 0$ such that

$$\|\mathbb{U}_k - \mathbb{U}_j\|_{L^\infty(\bar{\Omega})} \leq B$$

This implies that the sequence \mathbb{U}_j converges uniformly in $\bar{\Omega}$. Let us put $\mathbb{U} = \lim \mathbb{U}_j$, it is clear that $\mathbb{U} \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega})$, $\mathbb{U} = \varphi$ on $\partial\Omega$. Moreover, $(dd^c\mathbb{U}_j)^n$ converges to $(dd^c\mathbb{U})^n$ in the sense of currents, then $(dd^c\mathbb{U})^n = f\beta^n$ in Ω . The uniqueness of the solution comes from the comparison principle (see [BT76]). \square

Our next step is to construct Hölder continuous sub-barrier and super-barrier to the Dirichlet problem when $f \in L^p(\Omega)$ for some $p > 1$ and $\varphi \in \mathcal{C}^{0,1}(\partial\Omega)$.

Proposition 5.4. *Let $\varphi \in \mathcal{C}^{0,1}(\partial\Omega)$ and $0 \leq f \in L^p(\Omega)$, for some $p > 1$. Then there exist $v, w \in PSH(\Omega) \cap \mathcal{C}^{0,\alpha}(\bar{\Omega})$ where $\alpha < 1/(nq + 1)$ such that $v = \varphi = -w$ on $\partial\Omega$ and $v \leq \mathbb{U} \leq -w$ on Ω .*

Proof. Fix a large ball $B \subset \mathbb{C}^n$ so that $\Omega \Subset B \subset \mathbb{C}^n$. Let \tilde{f} be a trivial extension of f to B . Since $\tilde{f} \in L^p(\Omega)$ is bounded near ∂B , the solution h_1 to $Dir(B, 0, \tilde{f})$ is Hölder continuous on \bar{B} with exponent $\alpha_1 < 2/(nq + 1)$ (see [GKZ08]). Now let h_2 denote the solution to the Dirichlet problem in Ω with boundary values $\varphi - h_1$ and the zero density. Thanks to Theorem A, we see that $h_2 \in \mathcal{C}^{0,\alpha_2}(\bar{\Omega})$ where $\alpha_2 = \alpha_1/2$. Therefore, the required barrier will be $v = h_1 + h_2$. It is clear that $v \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega})$, $v|_{\partial\Omega} = \varphi$ and $(dd^c v)^n \geq f\beta^n$ in the weak sense in Ω . Hence, by the comparison principle we get that $v \leq \mathbb{U}$ in Ω and $v = \mathbb{U} = \varphi$ on $\partial\Omega$. Moreover we have $v \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$ for any $\alpha < 1/(nq + 1)$.

Finally, it is enough to set $w = \mathbb{U}(\Omega, -\varphi, 0)$ to obtain a super-barrier to the Dirichlet problem $Dir(\Omega, \varphi, f)$. We note that $w \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega})$, $-w = \varphi$ on $\partial\Omega$ and $\mathbb{U} \leq -w$ on $\bar{\Omega}$. Furthermore, by Theorem A, $w \in \mathcal{C}^{0,1/2}(\bar{\Omega})$ and then $w \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$ for any $\alpha < 1/(nq + 1)$. \square

When $f \in L^p(\Omega)$ for $p \geq 2$, we are able to find a Hölder continuous barrier to the Dirichlet problem with more better Hölder exponent. The following theorem was proved in [Ch14] for the complex Hessian equation and it is enough here to put $m = n$ for the complex Monge-Ampère equation.

Theorem 5.5. ([Ch14]). *Let $\varphi \in \mathcal{C}^{0,1}(\partial\Omega)$ and $0 \leq f \in L^p(\Omega)$, $p \geq 2$. Then there exist $v, w \in PSH(\Omega) \cap \mathcal{C}^{0,1/2}(\bar{\Omega})$ such that $v = \varphi = -w$ on $\partial\Omega$ and $v \leq \mathbb{U} \leq -w$ in Ω .*

Now we recall the comparison principle for the total mass of laplacian of plurisubharmonic functions.

Lemma 5.6. *Let $u, v \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega})$ such that $v \leq u$ on Ω and $u = v$ on $\partial\Omega$. Then*

$$\int_{\Omega} dd^c u \wedge \beta^{n-1} \leq \int_{\Omega} dd^c v \wedge \beta^{n-1}.$$

Proof. First assume that there exists an open set $V \Subset \Omega$ such that $u = v$ on $\bar{\Omega} \setminus V$. Let $h \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega})$ such that $h = 0$ on $\partial\Omega$. Then integration by parts yields

$$\int_{\Omega} h dd^c(v - u) \wedge \beta^{n-1} = \int_{\Omega} (v - u) dd^c h \wedge \beta^{n-1}.$$

Let V_1 be an open set such that $V \Subset V_1 \Subset \Omega$ and define the function $h = \max(-1, \rho/m)$ where ρ be the defining function of Ω and $m = |\sup_{\partial V_1} \rho|$. It is clear that $h \in PSH(\Omega) \cap \mathcal{C}(\bar{\Omega})$, $h = 0$ on $\partial\Omega$ and $h = -1$ on \bar{V}_1 . Since $u = v$ on $\bar{\Omega} \setminus V$, we get

$$\int_{\Omega} dd^c(v - u) \wedge \beta^{n-1} = \int_{V_1} dd^c(v - u) \wedge \beta^{n-1}.$$

We note that

$$\begin{aligned}
\int_{V_1} dd^c(v-u) \wedge \beta^{n-1} &= - \int_{V_1} hdd^c(v-u) \wedge \beta^{n-1} \\
&= - \int_{\Omega} hdd^c(v-u) \wedge \beta^{n-1} \\
&= - \int_{\Omega} (v-u)dd^c h \wedge \beta^{n-1} \geq 0.
\end{aligned}$$

Hence we obtain

$$\int_{\Omega} dd^c u \wedge \beta^{n-1} \leq \int_{\Omega} dd^c v \wedge \beta^{n-1}.$$

Now if we have only $u = v$ on $\partial\Omega$, then we define for small $\epsilon > 0$, the function $u_{\epsilon} := \max(u - \epsilon, v)$. Then we see that $v \leq u_{\epsilon}$ on Ω and $u_{\epsilon} = v$ near the boundary of Ω .

Therefore, we have

$$\int_{\Omega} dd^c u_{\epsilon} \wedge \beta^{n-1} \leq \int_{\Omega} dd^c v \wedge \beta^{n-1}.$$

We know by the convergence's theorem of Bedford and Taylor that $dd^c u_{\epsilon} \beta^{n-1} \rightharpoonup dd^c u \wedge \beta^{n-1}$ when $\epsilon \searrow 0$. Thus we have

$$\int_{\Omega} dd^c u \wedge \beta^{n-1} \leq \int_{\Omega} dd^c v \wedge \beta^{n-1}.$$

which proves the required inequality. \square

5.1. Proof Theorem B. Let U_0 the solution to the Dirichlet problem $Dir(\Omega, 0, f)$. We first claim that the total mass of ΔU_0 is finite in Ω . Indeed, let ρ be the defining function of Ω , then by Corollary 5.6 in [Ce04] we get that

$$\begin{aligned}
(5.2) \quad \int_{\Omega} dd^c U_0 \wedge (dd^c \rho)^{n-1} &\leq \left(\int_{\Omega} (dd^c U_0)^n \right)^{1/n} \cdot \left(\int_{\Omega} (dd^c \rho)^n \right)^{(n-1)/n} \\
&\leq \left(\int_{\Omega} f \beta^n \right)^{1/n} \cdot \left(\int_{\Omega} (dd^c \rho)^n \right)^{(n-1)/n}.
\end{aligned}$$

Since Ω is a bounded SHL domain, there exists a constant $c > 0$ such that $dd^c \rho \geq c\beta$ in Ω . Hence the inequality 5.2 yields

$$\begin{aligned}
\int_{\Omega} dd^c U_0 \wedge \beta^{n-1} &\leq \frac{1}{c^{n-1}} \int_{\Omega} dd^c U_0 \wedge (dd^c \rho)^{n-1} \\
&\leq \frac{1}{c^{n-1}} \left(\int_{\Omega} f \beta^n \right)^{1/n} \cdot \left(\int_{\Omega} (dd^c \rho)^n \right)^{(n-1)/n}
\end{aligned}$$

Now we note that the total mass of complex Monge-Ampere measure of ρ is finite in Ω by Chern-Levine-Nirenberg inequality since ρ is psh and bounded in a neighborhood of $\bar{\Omega}$ (see [BT76]). Therefore, the total mass of ΔU_0 is finite in Ω .

Let $\tilde{\varphi}$ be a $\mathcal{C}^{1,1}$ -extension of φ to $\bar{\Omega}$ such that $\|\tilde{\varphi}\|_{\mathcal{C}^{1,1}(\bar{\Omega})} \leq C\|\varphi\|_{\mathcal{C}^{1,1}(\partial\Omega)}$ for some $C > 0$. Now, let $v = A\rho + \tilde{\varphi} + U_0$ where $A \gg 1$ such that $A\rho + \tilde{\varphi} \in PSH(\Omega)$. By the comparison principle we see that $v \leq U$ in Ω and $v = U = \varphi$ on $\partial\Omega$. Since ρ is psh in a neighborhood of $\bar{\Omega}$ and $\|\Delta U_0\|_{\Omega} < +\infty$, we get that $\|\Delta v\|_{\Omega} < +\infty$. Then by Lemma 5.6 we have $\|\Delta U\| < +\infty$.

The Proposition 5.4 gives the existence of Hölder continuous barriers to the Dirichlet problem. Then using Theorem 5.2 we obtain the final result that is when $f \in L^p(\Omega)$ for some $p > 1$, we get $U \in PSH(\Omega) \cap \mathcal{C}^{0,\alpha}(\bar{\Omega})$ where $\alpha < 1/(nq + 1)$.

Moreover, if $f \in L^p(\Omega)$ for some $p \geq 2$, we can get better result. By Theorem 5.5 and Theorem 5.2, we see that $U \in PSH(\Omega) \cap \mathcal{C}^{0,\alpha}(\bar{\Omega})$ where $\alpha < \min\{1/2, 2/(nq + 1)\}$.

Remark 5.7. It is shown in [GKZ08] that we cannot expect a better Hölder exponent than $2/nq$ (see also [Pl05]).

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